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Translated by N. D. Friedman



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# Generalization of a Method of Determining the Stability of Arbitrarily Arranged Vortex trails

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In the present note we give a simple and general derivation of the necessary conditions for equilibrium of vortex trails with arbitrary mutual position of two of its parallel chains. These conditions permit one to judge the stability or instability of each of three possible arrangements: a) symmetrical trails; b) trails with staggered vortex positions; and c) asymmetric trails. To this end, on the one hand, we use the method of Kochin for small displacement in the form that he assumed for cases (a) and (b) and on the other hand, we use the method of solving differential equations for the perturbed motion of vortices set up in our work [2].

Resorting, as usual, to the alternate considerations of vortex systems (vortices with even index in one chain have the same displacement and vortices with odd index another, whereby for each of the two chains these displacements are different, therefore, two adjacent vortices have different displacements) and introducing, correspondingly, the displacements $\delta z_1'$, $\delta z_2'$, $\delta z_1''$, $\delta z_2''$ (the prime refers to the upper chain and the double prime to the lower) which, in particular, obey the conditions $\delta z_2' = -\delta z_1'$; $\delta z_2'' = -\delta z_1''$

Kochin obtains the system:

$$\frac{d}{dt}(\delta \bar{z}_1') = -\frac{\Gamma\pi}{8\,l^2 i}(A\,\delta z_1' + B\,\delta z_1'')$$
$$\frac{d}{dt}(\delta \bar{z}_1'') = \frac{\Gamma\pi}{8\,l^2 i}(B\,\delta z_1' + A\,\delta z_1'') \qquad (1)$$

where

$$A = 2 - 1/\sin^2\alpha - 1/\cos^2\alpha \qquad (2)$$
$$B = 1/\sin^2\alpha - 1/\cos^2\alpha$$

and

$$\alpha = \frac{\pi}{2}(\lambda + i\varkappa); \quad \lambda = d/l \;; \; \varkappa = h/l \qquad (3)$$

Here, $2h$ is the width of the trail, $2l$ is the distance between two adjacent vortices in the same chain and $2d$ is the lag of one vortex chain with respect to the other. We have: for $d = 0$ ($\lambda = 0$), the vortex chains are exactly one below the other (symmetrical arrangement); for $d = l/2$ ($\lambda = \frac{1}{2}$) they lag each other symmetrically by $l$ (staggered position); for $d < l/2$ ($\lambda < \frac{1}{2}$) they lag by an amount less than $l$ (asymmetric arrangement). For brevity, let us put in (1) $q = \Gamma\pi/8l^2$ and

$$\delta z_1' = \xi' + i\eta' = Z'; \qquad \delta z_1'' = \xi'' + i\eta'' = Z'' \qquad (4)$$

Then the system (1) becomes

$$\frac{d\bar{Z}'}{dt} = qi(AZ' + BZ'')$$
$$\frac{d\bar{Z}''}{dt} = -qi(BZ' + AZ'') \qquad (5)$$

If we add and subtract the left and right side of (5) we obtain

$$\frac{d}{dt}(\bar{Z}' + \bar{Z}'') = qi(A-B)(Z' - Z'')$$
$$\frac{d}{dt}(\bar{Z}' - \bar{Z}'') = qi(A+B)(Z' + Z'') \qquad (6)$$

But from (2) we find

$$A + B = -2\tan^2\alpha \;;\; A - B = -2\cot^2\alpha \tag{7}$$

and from (4) we obtain

$$\begin{aligned} Z' + Z'' &= (\xi' + \xi'') + i(\eta' + \eta'') = X + iY = Z \\ Z' - Z'' &= (\xi' - \xi'') + i(\eta' - \eta'') = \bar{X} + i\bar{Y} = W \\ \bar{Z}' + \bar{Z}'' &= (\xi' + \xi'') - i(\eta' + \eta'') = X - iY = \bar{Z} \\ \bar{Z}' - \bar{Z}'' &= (\xi' - \xi'') - i(\eta' - \eta'') = \bar{\bar{X}} - i\bar{Y} = \bar{W} \end{aligned} \tag{8}$$

Now (6) reduces to

$$\frac{d\bar{Z}}{dt} = qi(A - B)W = \bar{q}i\,\frac{W}{T}$$

$$\frac{d\bar{W}}{dt} = qi(A + B)Z = \bar{q}iTZ \qquad (\bar{q} = -2q) \tag{9}$$

where $T = \tan^2\alpha$ .

In order to be able to represent (9) in real form with the help of a system of fonr differential equations for the displacement, we write $T$ in the form

$$T = P + iQ; \quad \frac{1}{T} = \bar{P} + i\bar{Q} = \frac{P - iQ}{P^2 + Q^2} \tag{10}$$

and use the value of $\alpha$ from (3). We have

$$\tan\alpha = \tfrac{1}{2}\,\frac{\sin\lambda\pi + i\sinh\kappa\pi}{\cos^2\tfrac{1}{2}\lambda\pi + \sinh^2\tfrac{1}{2}\kappa\pi} \tag{11}$$

Finally, we obtain

$$P = \tfrac{1}{4}\,\frac{\sin^2\lambda\pi - \sinh^2\kappa\pi}{(\cos^2\tfrac{1}{2}\lambda\pi + \sinh^2\tfrac{1}{2}\kappa\pi)^2} \;;\quad Q = \tfrac{1}{2}\,\frac{\sin\lambda\pi\sinh\kappa\pi}{(\cos^2\tfrac{1}{2}\lambda\pi + \sinh^2\tfrac{1}{2}\kappa\pi)^2} \tag{12}$$

$$\bar{P} = P/\,P^2 + Q^2 \;;\quad \bar{Q} = -\,Q/P^2 + Q^2 \tag{13}$$

With the aid of (9) and (8) we can write

$$\frac{d}{dt}(X - iY) = \bar{q}i(\bar{P} + i\bar{Q})(\bar{X} + i\bar{Y})$$
$$\frac{d}{dt}(\bar{X} - i\bar{Y}) = \bar{q}i(P + iQ)(X + iY) \tag{14}$$

whence the real system has the form

$$\frac{dX}{dt} = \bar{q}(\overline{PX} - \overline{QY}) \ ; \quad \frac{dY}{dt} = -\bar{q}(\overline{PY} + \overline{QX})$$
$$\frac{d\bar{X}}{dt} = q(PX - QY) \ ; \quad \frac{d\bar{Y}}{dt} = -q(PY + QX) \tag{15}$$

Assuming for the particular solution of the above-mentioned system the expressions $X = Me^{\omega t}$, $\bar{X} = Ne^{\omega t}$, $Y = Re^{\omega t}$, $\bar{Y} = Se^{\omega t}$ we find an algebraic system of equations, homogeneous with respect to M, N, R, and S:

$$\begin{aligned} M - \bar{q}\bar{P}N + \bar{q}\,\bar{Q}S &= 0 \\ R + \bar{q}\bar{P}S + \bar{q}\,\bar{Q}N &= 0 \\ N - qPM + q\,QR &= 0 \\ S + qPQ + q\,QM &= 0 \end{aligned} \tag{16}$$

The common condition of this system leads us to

$$\omega^4 - 2\bar{q}^2(P\bar{P} + Q\bar{Q})\omega^2 + \bar{q}^4(P^2\bar{P}^2 + Q^2\bar{Q}^2 + \bar{P}^2Q^2 + P^2\bar{Q}^2) = 0 \tag{17}$$

the roots of which are

$$\omega_{1,2,3,4} = \pm\bar{q}\sqrt{P\bar{P} + Q\bar{Q} \pm i(\bar{P}Q - P\bar{Q})} \tag{18}$$

Substituting (13) in (18), we obtain for $\omega$

$$\omega = \pm\,\bar{q}\,\frac{P + iQ}{\sqrt{P^2 + Q^2}} = \pm\,K(P \pm iQ) \tag{19}$$

where $K = \bar{q}/\sqrt{P^2 + Q^2}$.

The solutions of the system of differential equations (15) with respect to the displacements $X$, $Y$, $\bar{X}$, $\bar{Y}$ have the form $e^{\pm KPt}\cos KQt$ and $e^{\pm KPt}\sin KQt$; with increasing t these displacements become infinite as a consequence of the presence of the positive exponent in the exponential. Therefore, whatever the position of the vortices in the trail, the latter in the case $P \neq 0$ is shown already unstable. For stability, it is necessary that $P=0$ . From (12) , in the same way, it follows that the greatest general necessary condition for stability is the condition

$$\sinh \varkappa\pi = \sin \lambda\pi \tag{20}$$

obtained by us earlier by other methods [3] .

Thus, we established that the asymmetric trails may also be stable, provided only that the parameters $\lambda$ and $\varkappa$ satisfy the relation (20), From condition (20), for $\lambda = \frac{1}{2}$, is obtained the already known condition for stability of a Karman staggered vortex street, namely, the necessary condition

$$\sinh \varkappa\pi = 1 \tag{21}$$

As concerns symmetric trails, for which $\lambda = 0$, then from (12) it follows that in this case $Q = 0$ and from (19) that $\omega = \pm 2\bar{q}$; the latter denotes the existence of a solution of the form $e^{-2\bar{q}t}$ $(\bar{q} < 0)$ is a fact corroborating the stability of the symmetric arrangement.

Returning to the stability of asymmetric trails, it is necessary here not to forget that, assuming the possibility of such an arrangement we assume the possibility of an oblique flow of vortex trails with which the axis of the vortex system maintaining a constant direction to the basic stream, moves parallel to itself, forsaking the position of the axis of symmetry of a streamlined body.

Actually, as is easy to conclude for the asymmetric arrangement the velocity components equal

$$U_A = \frac{\Gamma}{4l} \tanh \varkappa\pi \left[ \frac{1 + \tan^2 \lambda\pi}{\tan^2 \lambda\pi + \tanh^2 \varkappa\pi} \right]$$

$$V_A = - \frac{\Gamma}{4l} \tan \lambda\pi \left[ \frac{1 - \tanh^2 \varkappa\pi}{\tan^2 \lambda\pi + \tanh^2 \varkappa\pi} \right] \tag{22}$$

where $V_A \neq 0$ if $\lambda \neq 0, \frac{1}{2}$. For $\lambda = 0$( symmetric trails) we have from (22)

$$U_S = \frac{\Gamma}{4l} \coth \varkappa\pi \; ; \quad V_S = 0 \tag{23}$$

For $\lambda = \frac{1}{2}$ (staggered vortices)

$$U_Z = \frac{\Gamma}{4l} \tanh \varkappa\pi \; ; \quad V_Z = 0 \tag{24}$$

Let us add that on the basis of formulas (22), (23), (24), (20) and (21) it is easy to establish that this dependence

$$|\vec{W}_S| > |\vec{W}_A| > |\vec{W}_Z| \tag{25}$$

holds, i.e. that for determined $\varkappa$ , corresponding to any stable two-parameter trails the latter moves slower than the symmetric trails, but faster than the staggered. In conformity with the above, it is easy to obtain the property that for definite distances between vortices, $2l$, the asymmetric trail is always more stable than the staggered trail which appears at the last phase of every kind of equilibrium position. On the latter fact we already checked in our succeeding work. Here we only note that thanks to the existance of stable almost staggered vortices the presence is fully explained of such relations which although different from $\varkappa = h/l = 0.2805...$ is close to it. In conclusion, let us note that the exposition here of the theory of stability of generalized (two parameter) trails for small displacements is only approximate. As will be evident in later communications, those trails, similarly staggered, which Kochin considered, require revision in theory. We further show that asymmetric trails, fulfilling condition (20) may be considered as Kochin did, as the least unstable vortex mapping.

1. Kochin, Kibel, Roze: Theoretical Hydrodynamics, Chap. V, section 21 Pge 211, 1948

2. B. Dolapchiev: ZAMM, 17, (1937) Reprinted Bulgarian Academy of Science, 57, 149 (1938)

3. B. Dolapchiev: Annals of Sofia Univ., 39, 287 (1942)